Initial Value Problems

Suppose we have a evolution equation, time t is our independent variable and some quantity x is evolving in time.

An initial value problem looks like this:

This has infinitely many solutions. With a boundary condition, which is this case it the initial value of x at some time.

This problem has a analytical solution, as it is a linear ODE. But some ODEs are nonlinear:

Existence and Uniqueness: Under some assumptions there exists a unique solution to the ODE

Taylor Series Method

Recall that if a function has many derivatives at a point x, then we can approximate the function near x using those derivatives

For m = 1, we get Euler’s method, or time-stepping, the evaluate f at b, we take tiny steps from our initial value . Partitioning the interval using step sizes of , we get:

Local Error:

The global error is an accumulation of local errors: O(h)

What if m = 2? Then we can use the differential equation to find f’’.

Runge-Kutta Method

Idea:

The middle term is a differential operator.

RK2

The idea is to replace:

With a more complicated stepping function with judiciously chosen parameters:

Where

* Note that we don’t need to differentiate!
* Choosing gives us a good formula for approximating f.

RK4

Uses the exact same method, except discovered using a lot more math. The original paper is over 50 pages! Local error of , global error of .

Taylor Series Methods

Example 1:

Using the Taylor Series method, m=2

Where we use

Example 2:

Suppose we have a linear first-order ODE (this means if you have two solutions, any linear combination of them is a solution):

The solution of linear differential equations are exponentials:

Then

Example 3 (Stiff ODES):

Stiff ODEs occur when the function is almost constant, has very little variation along some interval.

Consider the case where . Then the exact solution is

Suppose we use Taylor Series to approximate the function at some point , n steps past our initial value.

In order for , we need .

To solve stiff ODEs one uses implicit methods:

The way this nonlinear equation is solved is using Newton’s Method.

In this example:

And in general

Adam-Bashforth Methods

The idea is to transform the differential equation in to an integral equation:

Higher Order ODEs

Now if we know how to solve first-order differential equation, how can we numerically solve a higher order ODE? We can transform it into a system of first of differential equations.

In mathematics you count things and try to see how thing changes

There is so so much mathematical knowledge I do not know.